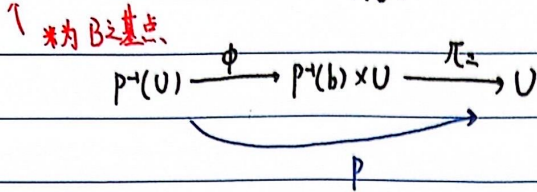
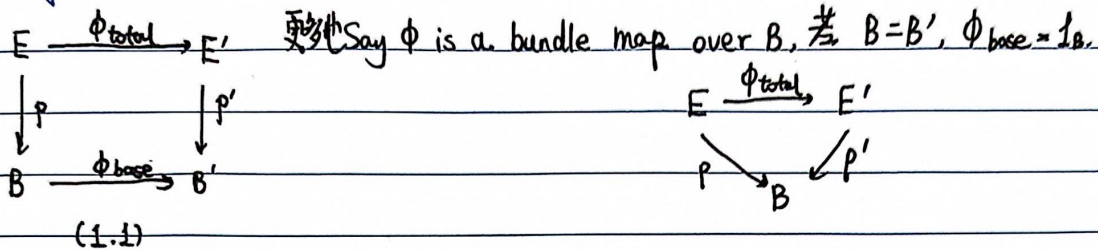


Fibre bundles, chapter 1.

DEF: B is a topological space with chosen base point $(*)$. 一个局部平凡的 B 上纤维丛是指 $E \xrightarrow{p} B$, E 被称为 total space, B 被称为 base space, 满足 $\forall b \in B, \exists$ an open neighborhood U of b , 且有在同胚 $\phi: p^{-1}(U) \rightarrow p^{-1}(*) \times U$. 使 $\pi_2 \circ \phi = p|_U$. (π_2 为 natural projection to the second factor).

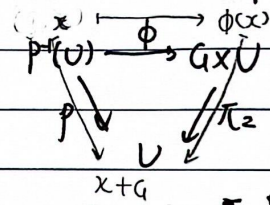


DEF: Morphisms of bundles: $(\phi_{total}, \phi_{base})$ 满足左图 (1.1)



DEF: principal bundle 一个主丛 over B 由一个纤维丛 $p: E \rightarrow B$, together with an action $G \curvearrowright E$ 组成, (G 为拓扑群) 且满足

- ① Shearing map $G \times E \rightarrow E \times E$ by: $(g, x) \mapsto (x, g \cdot x)$ 将 $G \times E$ 同胚地打倒 $E \times E$.
- ② $B = E/G, p: E \rightarrow E/G$ 为商映射
- ③ $\forall b \in B$, 存在开区域 U of b , 使 $p: p^{-1}(U) \rightarrow U$ is a G -bundle isomorphism to \mathbb{R}^n bundle $\pi: G \times U \rightarrow U$.



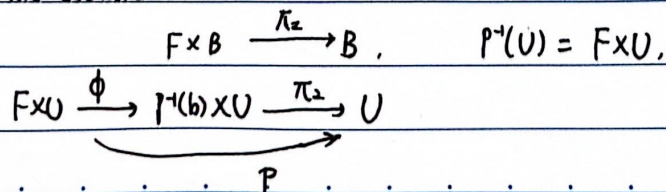
prop: (1) the shearing map is injective, 如果 \Leftrightarrow the group action is free ($g \cdot E$ 是一个完整的 copy of E)
这也说明 the action of G on total space of a principle bundle is always free.

(2). 一个自由作用产生了一个良定义的转移函数 $\tau: \mathcal{Q} \rightarrow G, \mathcal{Q} = \{(x, g \cdot x) \mid x \in X\}$ 为 shearing function 的象.

proof of (1): $(g_1, x), (g_2, x)$ maps to the same $(x, g \cdot x) \Leftrightarrow g_1 x = g_2 x = g x \Leftrightarrow$ the action is not free

(2) $(x, g \cdot x) \mapsto g$, 这是由于固定了 x 后, 群 G 中只有一个元素 g 能将 x 打倒 $g \cdot x$.

Example (1) $\forall F, B$, 均存在 trivial bundle $\pi_2: F \times B \rightarrow B$.



ω. M 为 n 维微分流形. $TM = \bigsqcup_{p \in M} T_p M$. $TM \rightarrow M$ 为一个 fibre bundle:

$p: TM \rightarrow M$ 如若 $v \in T_p M$, 取 M 的基点为原点 0 .

$b=q, U$ 为 q -邻域: $p^{-1}(U) \rightarrow p^{-1}(0) \times U$ 为一个同胚: $TU \rightarrow \phi(U) \times \mathbb{R}^n$

$v \mapsto (x^1(q), x^2(q), \dots, x^n(q), c^1(v), c^2(v), \dots, c^n(v))$

\cong
 U

\cong
 $p^{-1}(0)$ 为一个 n 维向量空间

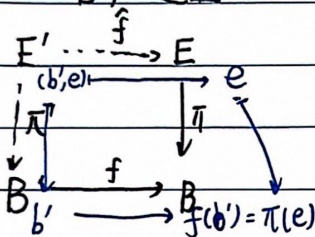
prop of bundle: $p: E \rightarrow B$ 为一个 bundle, 则 p 为一个开映射, 将开集映至开集

Operations on bundles

• pullback

若 $E \xrightarrow{\pi} B$ 为一个 bundle, 存在从 $B' \rightarrow B$ 的映射 f , 则我们可以将 $E \xrightarrow{\pi} B$ 这个 bundle 拉回至

$E' \rightarrow B'$, 这里 $E' = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\}$



• Cartesian product

(outer) $\xi_1: E_1 \xrightarrow{p_1} B_1, \xi_2: E_2 \xrightarrow{p_2} B_2$ 为两个 bundle, 定义 $\xi_1 \times \xi_2: E_1 \times E_2 \rightarrow B_1 \times B_2$ (外直积)

(inner): If ξ_1, ξ_2 are bundles over the same base B , 定义 internal Cartesian product of ξ_1 and $\xi_2: \Delta^*(\xi_1 \times \xi_2)$, 其中 $\Delta: B \rightarrow B \times B$ is the diagonal map

Vector bundles

本节中我们讨论 vector bundle 的一些额外性质