Fibre bundles, chapter 1.

DEF: B is a toplogical space with chosen base point (*). 一个局部平凡的 B 上之纤维丛是指 $E \xrightarrow{p} B$, E 被称为 total space, B 被称为 base space, 满足 $\forall b \in B$, $\exists$ an open neighborhood U of b, 且存在同胚 ϕ: $p^{-1}(U) \longrightarrow p^{-1}(*) \times U$, 使 $\pi_2 \circ \phi = p|_U$. (π_2 为 natural projection to the second factor).

*为 B 之基点.

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[rr, bend right, "p"'] & p^{-1}(b) \times U \arrow[r, "\pi_2"] & U \end{tikzcd}$$

DEF: Morphisms of bundles: $(\phi_{total}, \phi_{base})$ 满足左图 (1.1)

$$\begin{tikzcd} E \arrow[r, "\phi_{total}"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B \arrow[r, "\phi_{base}"] & B' \end{tikzcd}$$

(1.1)

更多地 Say ϕ is a bundle map over B, 若 $B = B'$, $\phi_{base} = 1_B$.

$$\begin{tikzcd} E \arrow[rr, "\phi_{total}"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

DEF: principal bundle 一个主丛 over B 由一个纤维丛 $p: E \longrightarrow B$, together with an action $G \curvearrowright E$ 组成. (G 为拓扑群) 且满足

① Shearing map $G \times E \longrightarrow E \times E$ by: $(g, x) \longmapsto (x, g.x)$ 将 $G \times E$ 同胚地打到 $E \times E$.

② $B = E/G$. $p: E \longrightarrow E/G$ 为商映射

③ $\forall b \in B$, 存在开区域 U of b, 使 $p: p^{-1}(U) \longrightarrow U$ is a G-bundle isomorphism to 平凡 bundle $\pi'': G \times U \longrightarrow U$.

$$\begin{tikzcd} x \arrow[r, mapsto] & \phi(x) \\ p^{-1}(U) \arrow[r, "\phi"] \arrow[dr, "p"'] & G \times U \arrow[d, "\pi_2"] \\ & U \end{tikzcd}$$

$x+G$

propsition: (1) the shearing map is injective, 如果 $\Leftrightarrow$ the group action is free ($g.E$ 是一个完整的 copy of E)

这也说明 the action of G on total space of a principle bundle is always free.

(2). 一个自由作用产生了一个良定义的转移函数 $\tau: Q \longrightarrow G$. $Q = \{(x, g.x)\} \subseteq X \times X$ 为 shearing function 的象.

proof of (1): (g_1, x), (g_2, x) maps to the same (x, gx) $\Leftrightarrow$ $g_1 x = g_2 x = gx$ $\Leftrightarrow$ the action is not free

(2) $(x, gx) \longmapsto g$, 这是由于固定了 x 后, 从群 G 中只有一个元素 (作用于x后) 能将 x 打到 gx.

Example (1) $\forall$ F, B, 均存在 trivial bundle $\pi_2: F \times B \longrightarrow B$.

$F \times B \xrightarrow{\pi_2} B$, $\quad p^{-1}(U) = F \times U$.

$$\begin{tikzcd} F \times U \arrow[r, "\phi"] \arrow[rr, bend right, "p"'] & p^{-1}(b) \times U \arrow[r, "\pi_2"] & U \end{tikzcd}$$

(2). M为n维微分流形. $TM = \coprod_{p\in M} T_pM$. $TM \longrightarrow M$为一个 fibre bundle:

$$\begin{aligned} p: TM &\longrightarrow M \\ v &\longmapsto p. \end{aligned}$$

如若 $v \in T_pM$. 取M的基点为原点0.

$b=q$, U为q之邻域 $\phi: p^{-1}(U) \longrightarrow p^{-1}(0)\times U$ 为一个同胚:

$$\begin{tikzcd} U \arrow[d, "\phi"'] \\ \phi(U)\subseteq \mathbb{R}^n \end{tikzcd}$$

$$\begin{aligned} TU &\longrightarrow \phi(U)\times \mathbb{R}^n \\ v &\longmapsto (\underbrace{x^1(q), x^2(q), \cdots x^n(q)}_{\cong\, U}, \underbrace{c^1(v), c^2(v) \cdots c^n(v)}_{\cong\, p^{-1}(0)}) \end{aligned}$$

$p^{-1}(0)$为一个n维向量空间

prop of bundle: $p: E \longrightarrow B$为一个bundle, 则 p为一个开映射, 将开集映至开集

Operations on bundles.

- pullback

若 $E \xrightarrow{\pi} B$ 为一个bundle, 存在从 $B' \longrightarrow B$的映射f, 则我们可以将 $E \xrightarrow{\pi} B$ 这个bundle拉回到

$E' \longrightarrow B'$, 这里 $E' = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\}$

$$\begin{tikzcd} E' \arrow[r, dashed, "\hat{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

$$\begin{tikzcd} (b', e) \arrow[r, mapsto] \arrow[d, mapsto] & e \arrow[d, mapsto] \\ b' \arrow[r] & f(b') = \pi(e) \end{tikzcd}$$

- Cartesian product

(outer) $\xi_1: E_1 \xrightarrow{p_1} B_1$, ξ_2 $E_2 \xrightarrow{p_2} B_2$ 为两个bundle, 定义 $\xi_1\times\xi_2$: $E_1\times E_2 \longrightarrow B_1\times B_2$. (外笛卡积)

(inner): If ξ_1, ξ_2 are bundles over the same base B. 定义 internal Cartesian product of ξ_1 and ξ_2: $\Delta^*(\xi_1\times\xi_2)$, 其中 $\Delta: B \longrightarrow B\times B$ is the diagonal map

Vector bundles

本节中我们讨论 vector bundle 的一些额外性质.